The Tower of Hanoi and the Experience of Lived Number

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As the study of experience, phenomenology lays claim to subject matter that is of great interest in both e-learning and technical design: this is the *experience* of users or of students in their various engagements with technology. The term "experience" has recently played a prominent role in discussions of computer technologies and other topics, with release in recent years of books on "user experience design" (e.g., Press & Cooper, 2003) and "experiential marketing" (e.g., Lenderman, 2005). In their book *Technology as Experience*, McCarthy and Wright (2004) explain the prominence of this term in the context of the recent "explosion" of new popular technologies and innovations:

> Interaction with technology is now as much about what people feel as it is about what people do. It is as much about children playing with GameBoys, teenagers gender swapping, and elderly people socializing on the Internet as it is about middle-aged executives managing knowledge assets, office workers making photocopies, or ambulance controllers dispatching ambulances. The emergence of the computer as a consumer product has been accompanied by very explicit attention to user experience. (p. 9)

Technologies and their design require new ways of understanding and studying experiential vicissitudes bound up with their uses. McCarthy and Wright and before them, authors like Paul Dourish, Terry Winograd, and Fernando Flores, make significant use of hermeneutic phenomenology and related "experiential" approaches to provide these kinds of studies of technologies.

In the case of e-learning research, attention to experience has taken a rather different character, but one that is no less forceful and explicit. In this research context, "experience"—above all student experience—tends to be understood in a number of predetermined ways: in terms of automatically generated records of online student activity and interactivity (e.g., Heckman and Annabi, 2003); in terms of levels of student satisfaction (e.g., Chiu, Stewart, & Ehlert, 2003); and student performance and attrition (e.g., Picciano, 2002; see also the review of Hiltz & Shea, 2005, pp. 149–156). Research on student experience, thus understood, ranges from informal reflections on "what happened" in this or that online class to formal studies using highly standardized methodological instruments and measures. For example, one study speaks of "instructors and administrators" being able to use a set of standards to "measure various aspects of the distance education experience and their importance to students" (e.g., Jurczyk, Kushner-Benson, & Savery, 2004). These "aspects of experience" are "measured" through the use of questionnaires that present students with preset questions and question types, including multiple choice, Likert scale, sentence completion, ordinal ranking, and others (e.g., Burgess, 2001, pp. 8–10). Questions and topics for student responses in such questionnaires are determined, of course, in advance. Student experiences, correspondingly, are predefined in terms of degrees of favorable or unfavorable responses or in terms of predetermined characteristics of a course or program, its delivery, and measures of its ultimate outcomes.

Of course, what gets lost in these categories, recordings, and measures is the vivid, concrete, situated, and irreplaceable character of experience, and the fact that it is "felt" and "lived," rather than something made available for detached analysis. Existence and experience are always primary to categories and measures. In their investigation of "technology as experience," McCarthy and Wright emphasize this same point, quoting anthropologist Clifford Geertz (1986) on the centrality of experience in social and cultural forms of analysis: "...without [experience] or something like it analyses seem to float several feet above their human ground" (p. 374). Analysis must instead "engage some sort of felt life, which might as well be called experience" (p. 374). McCarthy and Wright (2004) extend this argument to the study of technological experience:

> By excluding or separating off people's felt experiences with technology...people's concerns, enthusiasms, and ambivalence about participation are abstracted away or averaged out. ...If we are not to hover above the human ground, we must engage with the felt life, "which might as well be called experience." (p. 49)

Students and instructors undertake their learning and teaching in concrete situations and necessarily take up unique positions from which they are able to speak and give voice to their experience. It is up to the researcher to listen and to initially hold theory and categorization at bay, in order to engage the felt life in which this student or that teacher is "always-already" immersed.

It is consequently the goal of this chapter to engage this felt life and to try to get at a small part of the "human ground" of student experience in online contexts. It will do so through the following steps:

1. By first reviewing a few points from the previous chapter in order to explain how the data in this study was gathered and how the vital collaborative processes of writing, reading, and rewriting were undertaken.
2. By then presenting to the reader the text of the hermeneutic-phenomenological study itself, written in the manner of the Utrecht School. This study starts with the historical and experiential and moves to an exploration of concrete exercises and more abstract principles in computer science and especially in mathematics.
3. Finally, by moving from from mathematics back to experience, showing how "felt life" interpenetrates mathematical understanding and how this interconnection finds confirmation in quotations and characterizations of mathematicians themselves.

In keeping with the heuristics and guidelines for the hermeneutic-phenomenological method described in the previous chapter, the experiential data used in this study are acquired through conversational and relatively informal means and are interpreted and developed through highly collaborative research activity. The data was gathered specifically from school-age children and was further developed and integrated into a larger descriptive whole through the close collaboration of the co-authors. This collaboration began with discussions of methodology and mathematics education and continued through development of "lived experience descriptions" and their refinement into written "anecdotal" descriptions. The collaborative development and integration of these descriptive passages continued into the later stages of interpretation, presentation, writing, and rewriting. The process of shaping and refining the chapter, finally, occurred not just through the close collaboration of the authors. It also involved significant interchange among a number of researchers. This occurred principally in the context of a number of international conferences and workshops at which the chapter was presented in draft form. This allowed feedback and responses to be gathered from a number of different audiences; and these responses, in turn, served as the basis for further rewriting and revision.

As described in the previous chapter, the two processes—of collaborative writing (and reading and rewriting) and of more general "public" feedback and revision—are an integral part of the research process in hermeneutic phenomenology. These are among the most important ways through which experiential data and their presentation and amplification are validated. These processes, of course, are not determined with any absolute finality at the end of any one reading, presentation, revision, or rereading. Instead, they are enacted or undertaken anew each time the text is encountered. It is in this sense that a hermeneutic-phenomenological study and the descriptive writing that is at its core is a *process* more than it is a *product*. And this process, the authors hope, continues as new readers encounter the study presented here.

Just as any research report will define its investigation in terms of a particular research question, the study presented in this chapter also has a specific question at its core. In hermeneutic phenomenological research, however, such a question has particular characteristics. "The essence of the *question,*" as Hans-Georg Gadamer (1989) puts it, "is to open up possibilities and keep them open" (p. 299; emphasis in original). A question of this kind, in other words, should not be framed or articulated in such a way that it unnecessarily forecloses on certain answers or contains a predetermined solution: "the openness of what is in question consists in the fact that the answer is not settled," as Gadamer explains (p. 363). The point of phenomenological research, after all, is not to define and solve problems, but to elicit new, experientially rich ways of looking at things from the perspective of its readers, by cultivating not a sense of certainty and finality, but of wonder.

The question posed by the hermeneutic-phenomenological study taken up in this chapter is: "What is experience of engagement with the Tower of Hanoi puzzle?" The Tower of Hanoi, of course, is a famous child's toy or puzzle and its characteristics will be described in some detail below. For now, it is important to note that the question about engaging with this puzzle is further qualified in this study in a number of ways: The focus is, where possible, on *children's* experience of the puzzle, on encounters with it as it has been re-created as an interactive puzzle *online* and finally, on its significance in the context of *mathematics* education.

Finally, in considering the question of the experience of engaging with the Tower of Hanoi puzzle, it should be mentioned that the study presented here makes use of a number of specific techniques associated with hermeneutic phenomenology, as presented earlier. These include anecdotal description, which in this study is presented as a set of linked narratives, and reflections, told in the first person. It also includes the use of the four existential dimensions mentioned previuosly: lived space, lived body, lived time, and lived human relation. These dimensions or "existentials," as the reader will see, are invoked in interactions or "interviews" with children and also constitute an important part of the study's tentative conclusions about the engagement with the Tower of Hanoi puzzle and mathematics education. This study also makes use of a range of sources to develop its understanding of experiences in engaging with the Tower of Hanoi puzzle. These include linguistic sources, specifically the terms or language that are and have been used to describe the Tower of Hanoi itself and other puzzles like it. These sources also include the research and teaching literatures of psychology, math education, and computer science. These do not simply serve as examples of theories that must be bracketed in order to gain less indirect access to experience; they are also utilized as indicators of the broad, even mysterious, appeal of puzzles like the Tower of Hanoi.

History of the Tower of Hanoi

But this study does not begin with reference to computer science, psychology, or math education. It begins instead with history, which is also a potentially rich source of information and inspiration, and a source with which the Tower of Hanoi is deeply intertwined. This history appears to have begun in 1550, when the Italian mathematician Girolamo Cardano is said to have written the following about the then mysterious lands of the Far East:

> A monastery in Hanoi has a golden board with three wooden pegs on it. The first of the pegs holds sixty-four gold disks in descending order of size—the largest at the bottom, the smallest at the top. The monks have orders from God to move all the disks to the third peg while keeping them in descending order, one at a time. A larger disk must never sit on a smaller one. All three

> pegs can be used. When the monks move the last disk, the world would end. (As quoted in Danesi, 2004, pp. 109–110)

Other versions of this story talk of diamond-tipped poles; still others speak of moving the disks between three different holy places. But in each case, this sacred duty involves the measurement of an immeasurable, unimaginable period of time. The specific task of moving a disk between poles or locations seems to have the same basic function in each: namely to break into discrete tasks and moments an interval of time that—because of its enormous size or length—is difficult, perhaps impossible to comprehend. And this is an interval of time that is of special significance: the endpoint of this interval also marks the close of human time itself.

The Tower enters modern history in 1883, with mathematician François Anatole Lucas. This was the year that Lucas introduced to the market a simplified, abstracted version of the Tower in the form of a *casse-tête*, a puzzle, brain teaser, or literally, a "head breaker." Similar to but simpler than the situation described above by Cardano, Lucas' version has three pegs and only eight disks. But the same rules and restrictions apply. Lucas promoted the puzzle as follows:

> Amusing and instructive, easy to learn and to play in town, in the country, or on a voyage, it has for its aim the popularization of science, like all the other curious and novel games of professor N. CLAUS (OF SIAM). (As translated by Stockmeyer, 1998)

Referring to himself with this vaguely Asiatic, anagrammatic pseudonym of N. Claus (of Siam), Lucas goes on to make reference to the original legend of the Far East with its 64 discs. Clearly aware of a rather precise, mathematical relationship between the numbers of disks and the moves required, Lucas offered his customers this challenge:

> We can offer a prize of ten thousand francs, of a hundred thousand francs, of a million francs, and more, to anyone who accomplishes, by hand, the moving of the Tower of Hanoi with sixty-four levels, following the rules of the game. (As translated by Stockmeyer, 1998)

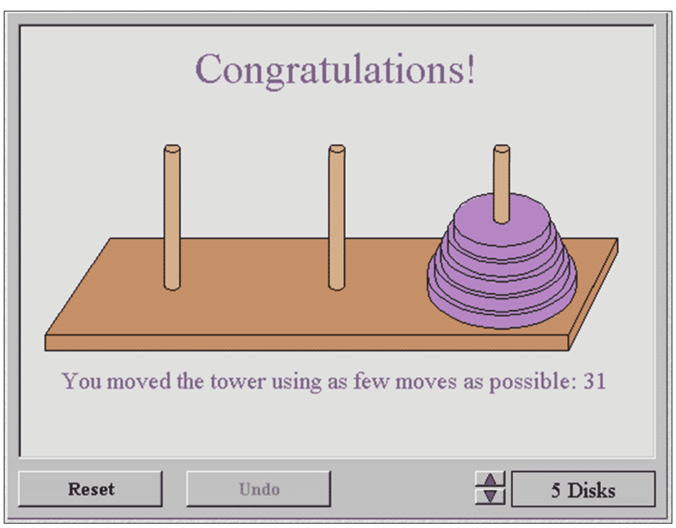
Underneath, he knowingly added: "We will say immediately that it would be necessary to perform successively a number of moves equal to 18 446 744 073 709 551 615 which would require more than five billion centuries!” (As translated by Stockmeyer, 1998).

“Virtual” Towers

The Tower of Hanoi, as will be discussed below, stands as a kind of “textbook” problem in computing science and artificial intelligence. In addition, it serves as a kind of paradigmatic mental or problem-solving “task” in present-day psychology and psychiatry. Also, computerized versions of this puzzle, like the one shown below, now abound on the Web (which is where I first discovered the Tower of Hanoi).

Figure 7.1:
Tower of Hanoi
From the National
Library of Virtual
Manipulatives ,
Utah State University
http://nlvm.usu.edu
/en/nav/frames_asid
_118_g_3_t_2.html

The game, like the one shown above, initially presented seven purple disks stacked on the left-most peg. At first glance, the puzzle looked quite innocuous. Within moments I was absorbed in the task of trying to move the disks over. I quickly found myself frustrated as the disks fell out of their sequence. Pausing momentarily to decide how to put them back in order, I began to feel like a bit like a dog chasing its tail. I had to stop and think: “Okay. I want to move the next big disk over. That means I need to first stack the others in the middle.” But moving the disks to accomplish this interim goal was difficult enough and I



was soon absorbed in this task alone. Still, the largest disks remained trapped in their original resting places, immobilized on the left side of the puzzle. Unable to bear the aggravation, and despite having the premonition of a headache coming on, I reset the game to three disks.

> Dana my youngest daughter looked at the puzzle briefly as she came by, showing little interest. She then started playing the piano beside me. She played "A Little Song" beautifully. I usually relish in her playing, but not at this moment.
>
> "Dana, do you have to play that piano now?" She stopped and disappeared from the room. I made a mental note to apologize to her later.
>
> I was quickly absorbed in the puzzle again. I accidentally moved the wrong disk and the machine counted my corrected move. That was unfair! I reset the puzzle and the pile of disks instantly appeared on the left peg. Finally, after a number of attempts, I was able to move the three disk Tower in 15 moves, the fewest possible.
>
> With a quick click of the mouse, I reset the game to four disks in order to challenge my new-found skills. I was able to move the Tower again, but not without error. The game flashed, "Congratulations! You moved the Tower in 39 moves. The Tower can be moved in fewer moves." What had begun as only the slightest twinge behind my forehead was quickly turning into a full-blown headache. "Darn it! Where did I go wrong?"

I work with these kinds of puzzles everyday. I judge a puzzle's merit by how much "brain teasing" the puzzle evokes. I dismiss puzzles that are solved too easily, without much effort. The puzzles that capture my thoughts and drive me crazy are always my favorites. I become very emotionally attached to these puzzles and solving one for the first time makes me feel ecstatic. It is reminiscient of the description of philosopher and mathematician Bertrand Russell (1945):

> To those of you who have reluctantly learned a little mathematics in school this may seem strange; but to those who have experienced the intoxicating delight of sudden understanding that mathematics gives from time to time, to those who love it…an element of ecstatic revelation will seem completely natural. (p. 33)

The puzzles that torture me the most are the ones I promote in my work with teachers and students. I visit school classrooms and use puzzles to engage or "tease" the brains of students and teachers alike. In these visits, we use "manipulables," "manipulatives," or objects that can be turned, flipped, and slid about. They are designed to help students understand mathematical abstractions and can range from popsicle sticks and building blocks to an abacus or a wooden version of puzzles like the Tower of Hanoi. An online puzzle like the one I tried above is generally called a "virtual manipulative." This type of manipulative has been defined as

> an interactive, Web-based visual representation of a dynamic object that presents opportunities for constructing mathematical knowledge... They are visual images on the computer that…can be manipulated in the same ways that a concrete manipulative can. (Moyer, Bolyard, & Spikell, 2002, p. 372)

But this still leaves one to wonder about the ways in which arrangements of toothpicks, blocks, or disks and pegs have to do with mathematics and equations. What's the mathematical lesson in a simple but enigmatic puzzle like the Tower of Hanoi?

The question of progressing from that which is manipulable and concrete to that which is abstract and formalized has been an important one in educational and developmental psychology. Developmentally, this progression is discussed in terms of stages or successive modes of intelligence. Jean Piaget, for example, described the passage from childhood to adolescence as coinciding with the progression between the "*concrete* operational" and the "*formal* operational," respectively (Ginsberg & Opper, 1979, pp. 153–155, 198–204). Also, Jerome Bruner (1966), influenced by Piaget, posited the existence of three successive modes of intelligence that he called the "enactive," "iconic," and "symbolic." The first of these—the enactive—focuses on the manipulation of concrete objects in the physical world, while the last of these—the symbolic—is exemplified by the manipulation of symbols in the mind (see pp. 10–11). In both cases, mental growth is understood in terms of a progression or movement *away* from the concrete and physical, and *toward* the formal, symbolic, and abstract.

More recently, Bransford, Brown, and Cocking (2000) have described a similar movement from the concrete to the abstract specifically in the context of mathematics education. Speaking particularly of teaching techniques in algebraic mathematics, these authors refer to this process as "progressive formalization":

> It begins by having students use their own words, pictures, or diagrams to describe mathematical situations to organize their own knowledge and work and to explain their strategies. In later units, students gradually begin to use symbols to describe situations, organize their mathematical work, or express their strategies. ...Later, students learn and use standard conventional algebraic notation for writing expressions and equations, for manipu-

> lating algebraic expressions and solving equations, and for graphing equations. (p. 137)

In this context, the Tower of Hanoi can be seen as helping students progress from "words, pictures, or diagrams" to more "standard conventional...notation." Students bring to the puzzle their existing knowledge of mathematics, of order, rules, games and concrete manipulation. They then use these to engage with the Tower of Hanoi in concrete and enactive terms. Guided by a tally of their moves and by feedback on the lowest possible number of moves, these predominantly concrete operations can be seen as leading the student to more "conventional," "formalized," and "symbolic" understandings. Students in this sense are able to build progressively on "informal ideas in a gradual but structured manner" (Bransford, Brown, & Cocking, 2000, p. 137). The intention is that these ideas can then be sufficiently refined to eventually be formalized, presumably and ultimately in the form of mathematical notations and expressions.

"It Made My Mind Angry"

But how is this related to the child's experience of engaging with the Tower of Hanoi?

> After I finally pulled myself away from the puzzle to finish my housework, I left the game up on the computer screen. A while later, I noticed that my daughter Dana had started to play with the Tower puzzle on the computer on her own. She sat with slouched shoulders, chin jutting out, staring intently at the screen. The disk wavered as she moved the mouse tentatively back and forth. Once in awhile, she had difficulty getting the disk to hold on the intended peg, but she soon got the hang of it. After several minutes she muttered "dang, I am going in circles." Then a few moments later she asks, "I'm not doing great, what is the high score?" "The point of the game is not to get a high score," I explained, "but to move all the disks in as few as moves as possible." She finished in 109 moves.
>
> Unhesitatingly, Dana changed the game to just two discs. She moved both disks in three moves. When the game congratulated her for moving the Tower in the minimum number of moves, she clapped her hands and exclaimed "Yeah! Yeah! Sweet!" She quickly worked her way up to four disks. She celebrated each achievement with as much enthusiasm as the first.

> Whenever the game told her she could have done it with less moves, she exclaimed “Dang!” or “Oh Crap!” And each time she determinedly started again.
>
> As I observed her having difficulty, I found myself wanting to offer unsolicited advice; when she succeeded, I found myself muttering sounds of agreement. “Please stop saying that,” she told me loudly, and then she turned back to the computer.
>
> Dana had successfully worked through moving five discs. However, she made a mistake while trying to pile them back on the sixth disc. She stared at the puzzle for what seemed like several minutes. Then a look of determination crossed her face and she clicked “reset” on the puzzle. She repeated this process for a full 45 minutes.
>
> When I heard her cry “Sweet!” some time later, I returned and asked her how many tries that took. “Billions of tries” she responded. I asked what she thought of the game. “Fun” was all she said. I asked if time passed quickly. She said, “No, it stood still. It felt like I had been playing for hours when I know it wasn’t that long.” I asked what she thought of playing a game that was at least 100 years old, maybe even centuries old. She said that people were nuts back then too and liked to torture themselves. The Tower of Hanoi had made her mind angry, she added.

What was it about the Tower that attracted us both in this strange way? At first, everything appears quite straightforward and innocuous; the puzzle is very simple on its own: There are only a few basic rules, three pegs, a few discs, and a tally of one’s moves. But gradually, the puzzle can draw those who engage with it into a strange world: it is a world where, as Dana says, time stands still. It feels like hours, when you know it isn’t that long. At the same time, the task being undertaken is very repetitive. Even when the puzzle is simplified to far fewer than 64 discs, it seems to take so many moves that they appear impossible to enumerate: Dana makes “billions of tries” and earlier, I myself was struggling to avoid unnecessary moves. The experiential world of this puzzle is also one that is sharply delineated from what is happening outside, be it music or even supportive feedback. In both cases, those nearby are harshly silenced. This seems to be consistent with how some mathematicians characterize their field of study. Stanislaw Ulam (1991), for example, refers to mathematics as “an escape from reality.” “The mathematician,” he says, “finds his own monastic niche and happiness in pursuits that are disconnected from external affairs” (p. 120).

Perhaps at first, both the Tower of Hanoi and mathematical experience generally seem closely connected with the cerebral, the mental, or the intellectual. As it might initially be understood in popular, mathematical, and psychological terms, this puzzle is about the head, the brain, or the intellect—not the heart, the body, or the emotions. Lucas called the puzzle a "*head* breaker," a "*brain* teaser"—something that "has for its aim the popularization of science." Indeed, in this sense, engagement with this puzzle has been fairly consistently presented as a formalized, disembodied, and thoroughly cerebral undertaking. As a virtual "manipulative," the Tower of Hanoi can certainly be seen as enabling a progressive, gradually increasing formalization in the learner's understanding. The number of moves is counted and sometimes with some outside encouragement (and certainly through the feedback offered by the virtual manipulative itself), users are led to strategize about the most efficient way to complete the *casse-tête*.

But of course, this is not the whole story. Dana's description of puzzle as making her "mind angry" confirms its cerebral nature but simultaneously points beyond it. The affective or physical nature of terms such as "teaser" or "breaker also seem to have this effect. Such words bring into play the emotions, the heart, and the realm of the somatic or the body. Elation and frustration are perhaps the most common feelings associated with the puzzle: both Dana and I felt and expressed jubilation after our initial success. Maybe we were sharing in something like "the intoxicating delight" or the "element of ecstatic revelation" that Bertrand Russell described as arising periodically in mathematics. I've also seen children's eyes well up with tears as they repeatedly run into difficulty in working with these kinds of puzzles. Just a few minutes later, though, the same pair of eyes can suddenly light up when the child finally finds a solution. Although a few people—especially adults, it seems—initially refuse to try the puzzle, once someone is engaged with it, their frustration generally does not seem to stop them from continuing and from trying again (and again). Rather than being satisfied with success when it is finally attained, however, those engaged with the puzzle seem to seek more frustration voluntarily, making the puzzle harder by adding more discs. For Dana and me, any sense of elation passed quickly as the Tower lured its "puzzlers" to continue.

The Tower of Hanoi in Scholarship

As indicated above, the role of the Tower of Hanoi in research is remarkably rich and complex. One could say that the Tower of Hanoi is a puzzle that has launched a thousand research projects. In psychology, for example, the puzzle has served as "a well-established test of executive [mental] functions" (Kopecky, Chang, Klorman, Thatcher, & Borgstedt, 2005, p. 625). It is used as a kind of paradigmatic "task" in measures of attentional and problem-solving ability. This "task" often serves an experimental control to test the effects of variables such as age, "divided attention," neuroses, psychoses, and other conditions on mental performance or problem-solving ability (e.g., Ronnlund, Lovden, & Nilsson, 2001; Vakil & Hoffman, 2004). It is also very familiar, as indicated above, in mathematics, and in research and teaching in both computer science and artificial intelligence. It is easy to find stratagems, algorithms, computer programs, and discussions of the complexities of "artificially intelligent" planning and problem solving that addresses the Tower of Hanoi in one form or another as a paradigmatic example. Speaking specifically of computer programming, one author explains,

> The Tower of Hanoi puzzle…has been undergoing a dramatic revival in popularity during the past years largely due to its use as a programming exercise in elementary computer courses. Many variations on the original puzzle also have been proposed and solved. (Stockmeyer et al., 1995, p. 37)

Many short programs or algorithms have also been developed to solve this puzzle. Diagrams and other formalized descriptions of different logical solutions for the Tower of Hanoi abound.

One example of a diagrammatic solution to the Tower of Hanoi is provided by the triangular "graph" below. It shows the possible, divergent paths for solving the puzzle working with three discs, starting from the top and proceeding to two possible "solutions" presented at its bottom corners. The pegs are designated by the letters a, b, and c. The specific order of these letters (provided in all of their possible variations at each "node" in the graph) indicates the location of the disks at a given stage.

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         ccc
          /\
    acc /__\ bcc
       /    \
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          abc /      \ bac
             /\      /\
        bbc /__\ __ /__\ aac
           / cbc    cac \
      bba /              \ aab
         /\              /\
    cba /__\aba      bab/__\ cab
       /    \          /    \
  caa /      \aca  bcb/      \ cbb
     /\      /\      /\      /\
aaa /__\ __ /__\ __ /__\ __ /__\ bbb
      baa bca cca  ccb acb abb
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(Tower of Hanoi, 2008)

An almost endless variety of other formalized solutions have been developed using computer languages of various kinds. These present simplified but unambiguous sets of coded instructions that a computer can execute in order to solve the puzzle. The example below presents a highly simplified English version of these kinds of instruction sets or algorithms, involving only three separate lines or directives:

1. Move the smallest disk to the peg it has not recently come from;
2. Move another disk legally (there will only be one possibility);
3. Repeat.

The reason that algorithms of this kind have been developed in such profusion is that they provide powerful illustration of a process known in computer science as "recursion": the repetition of a group of instructions as a set by that same group of instructions.

Exponential Experience

The examples provided above stand as different, formalized, symbolic representations of the logical problem presented by the Tower of Hanoi puzzle. In the language of Bransford, Brown, and Cocking, these diagrams and algorithms represent, in a very general sense, examples of formalized "conventional notation." Although they are not algebraic, these representations or formalisms present something that allows students of computer programming to work through their own process of "progressive formalization." Learning in this case can be understood as the process of moving from unformalized and possibly even idiosyncratic understandings to ones that can be expressed as elegantly and economically as in the diagrams, steps, and computer code provided above.

But this is obviously not all there is to the experience of the puzzle or to the related experiences of puzzlement and discovery. Mathematical and computational abstractions, or psychological and psychiatric categories, not surprisingly, do not begin to exhaust its experiential significance. Consider Dana's response to my promptings, below:

> After Dana had mastered six disks and she wanted to try seven, I asked her to predict how many moves that might take her. I wrote a list of the number of disks and corresponding numbers of moves.
>
> 2 disks – 3 moves
> 3 disks – 7 moves
> 4 disks – 15 moves
> 5 disks – 31 moves
>
> Then I asked her how many moves it would take for six disks. She looked at my list for just a second or two and said, "63 moves." Astonished, I asked, "How did you figure that out?" "The rule is itself plus itself plus one," she answered matter-of-factly. The total number of moves required increases consistently, in other words, by a factor of two. Dana, in other words, had informally stated an iterative pattern hidden in the Tower of Hanoi puzzle, not unlike the algorithmic diagrams and steps illustrated above.

I had been struggling myself to come to my own mathematical understanding of the puzzle. As a veteran of hundreds of puzzles and a researcher in mathematics education, I knew there was almost certainly a beguilingly concise and simple formula that would explain my observations and predict the number of moves for *any* number of disks. I thought about these two sets of numbers (for disks and moves) and about their seemingly infinite arrangements and looked for some kind of a pattern or clue.

Each time a single disk is added to the puzzle, the number of moves required to solve it increases. As the number of disks increases, the number of moves goes up from 1 to 3 to 7 to 15, and finally, to 31. When I added 1 to each of the numbers in this series, I was suddenly able to see what I had been looking for: 2, 4, 8, 16, 32. The new sequence gave me shivers up my spine; 2 times 2 equals 4, 2 times 2 times 2 equals 8, 2 multiplied by itself 4 times (2^4) equals 16. The relation of the number of disks to the number of moves can be expressed in terms of exponential values of 2 (with 1 subtracted at the end). You just have to calculate 2 to the power of the number of disks being used (and subtract 1), and *voila*, you have the total number of moves required. How simple, how elegant! Or to use Dana's term, how "sweet!"

In mathematical terms, I had just uncovered a geometric progression: a numerical sequence in which each term is multiplied by a constant in order to obtain the next term. Mathematically, the Tower is a model geometric progression that increases by exponential values of two. Each additional disk on the Tower, in short, doubles the number of moves required (with one subtracted from the final result). Of course, this can be expressed in the form of a mathematical equation, by establishing first that "y" is the number of moves, and "x" is the number of discs:

$$y = 2^x - 1$$

My surprise in making this discovery was perhaps not as powerful as the "ecstatic revelation" or the "intoxicating delight of sudden understanding" described by Russell. And of course, my breakthrough was hardly a new discovery for the field of mathematics. But another mathematician describes this experience in slightly different terms, making it clear that it *would* include my own pleasure and also the "tension" or "anger" that preceded it. In a commentary on "The Tears of Mathematics," Hungarian-born George Pólya writes

> there is a grain of discovery in the solution of any problem. Your problem may be modest, but if it challenges your curiosity and brings into play your inventive faculties, and if you solve it by your own means, you may experience the tension and enjoy the triumph of discovery. (Pólya, as quoted in Newman, 2000, p. 1978)

Understanding the Tower of Hanoi as a mathematical puzzle, in other words, is not just a matter of quantities and numerical magnitudes captured in conventional notation. Each attempt to solve the puzzle is not just a question of moving between nodes along graphed lines or following a set of simplified steps that can be executed in rapid recursion. The experience of engaging with the puzzle and in figuring out this solution could be characterized at once as an ordeal and a cause of vexation, or less often, as an occasion for relief or even elation and celebration. In this context, the difference in magnitude, say, between 2^2 and 2^6 is not just an indifferent, abstract quantity. Instead, one could say that there is, in engagement with the puzzle, a kind of "experiential" magnitude that is at least as significant.

This magnitude can be understood in terms of some of the life-world existentials of time, space, relation and body described earlier. The Tower of Hanoi puzzle, in this sense is experienced as "lived time" in the seemingly endless sequence of moves that each additional disk imposes on the player. This is an experiential temporality, as Dana observed, in which time "stands still," and in which minutes or seconds seem like hours. This is further reinforced in descriptions of the origins of the puzzle as a way of measuring an apparently unimaginable, immeasurable period of time. Such accounts present the puzzle as an experience of breaking down into discrete and comprehensible tasks and moments an immense interval of time that is difficult, perhaps impossible to comprehend. The Tower of Hanoi is also experienced as "lived space," in terms of a highly simplified life-world, created by the pegs and disks of the puzzle. This is one that separates the player from others in the "world" outside. This is a space or world, as observed above, where a set of simple rules can render some parts of the puzzle apparently immobile or impossible to release and move from one spot over to another. This movement from left to right, of course, often proves much more difficult as a whole than the actual, trivial physical distance that separates one pole from another would suggest.

What makes this experiential world significant *pedagogically* is the *kind* of mathematical relationship to which it can give such vivid experiential life. The explicit rules that must be followed in engagement with the Tower of Hanoi are on their own linear and serial (three pegs and two or more disks ordered by size). But the mathematics relevant to its actual *experience* is emphatically different. The mathematics that links the number of disks together with the number of turns is one that is unmistakeably geometric or *exponential* in nature. It is a mathematics that is illustrated, for example, in graphs showing the familiar curve of a parabola. This same, nonserial mathematics is referenced in measures of economic "growth rates" or of totals produced through compounded interest. But as is clear from these and other examples, this geometric relationship involves surprises or a kind of unexpected or uncontrolled character that is not present in linear or even in variable relationships or patterns. Think, for example, of increases of micro-processing power, doubling every eighteen months, resulting in disposable devices (e.g., musical greeting cards) that have more processing power than the first multimillion dollar main-frame computers. Think also of bacteria reproducing in the environment of a Petri dish, doubling every few hours, eventually exhausting the limited resources of this environment. Think perhaps more ominously of the potential spread of a virus like SARS, in which each carrier infects many others. The character of exponential relations is brought to life, of course, in a very different way in the Tower of Hanoi, but it is one that is marked by a singular experiential intensity.

This experiential intensity has significant implications for discussions of "progressive formalization" and of the developmental movement from the concrete to the symbolic described above. This experiential evidence suggests that as the student progresses toward formalization, he or she does not simply leave behind the nonsymbolic, the concrete, and the unformalized. The embodied operations of the concrete, of "enactive" engagement, in other words, are not simply overtaken or supplanted by the more advanced stages of the formal and symbolic. The concrete, unformed, emotional, and even physical aspects of engagement with mathematics continue to have a very significant role to play. They could hardly be more prominent and pronounced in the in the intoxication of discovery and in the tears or frustration of failure.

The ways in which mathematicians have described their experience with questions and problems as well as the "triumph" of their solution lends weight to this conclusion. For these descriptions of mathematical "tensions" and "elations" have strong somatic or corporeal connotations: Russell, for example, speaks of "intoxication" and uses the word "ecstasy"—derived from the Greek *ex-stasis*—suggesting being or standing outside oneself. Pólya similarly describes the embodied phenomena of the "tears" of mathematical "tension" and enjoyment of the "triumph" of mathematical discovery. Moreover, both Pólya's and Russell's descriptions make it clear that experiences that are charged with mathematical emotion are not some kind of extraneous distraction or curious side effect, but that they are at the very core of involvement with mathematics. These experiences are the "grain of mathematical discovery" (Pólya), a "completely natural" part of life for those who "love" mathematical endeavor (Russell). Even when encountered on a relatively simple level, the Tower of Hanoi can provide a foretaste of these kinds of mathematical emotions or experiences. It can provide a way of inaugurating the student into experiences that are at the very heart of mathematical problem solving.

In this sense, the Tower of Hanoi could also be understood as an emotional test or an "experience-able" rather than just a brain teaser or a "manipulative" (virtual or otherwise). The value and significance of the Tower of Hanoi is not exhausted as soon as it can be explained or re-presented through one graph or algorithm or one set of conventional notations or another. What is more important is the experience of discovery, frustration, and even obsession that the puzzle can provide, time and again, even after one has become "experienced" with it. It is through the emotionally charged, lived space and time opened up by this puzzle that it is possible to encounter what might even be called a sense of "lived number." And it is this experiential "vivification" that is of indispensable value in the Tower of Hanoi and possibly in other mathematics "manipulatives" as well.

Conclusion: The Effects of Experiential Research

In *Researching Lived Experience*, van Manen describes a number of ways in which hermeneutic-phenomenological studies and their results, can have varying real-world consequences or "effects." In keeping with the nature of this kind of research, these effects are generally not expressed in terms of "solutions" to preexisting "problems." At the same time, though, this does not mean that the effects are limited to the theoretical domain or the realm of "subjective" feeling. The intersubjective ground of phenomenological research, and of the writing and rewriting processes central to it, can ensure a broader significance than this.

The effects of a study such as the one presented in this chapter, for example, may be to awaken in its readers—whether they are math teachers or students or from a different educational background altogether—a sense of the expanded and even inexhaustible significance of a simple mathematics manipulative such as the Tower of Hanoi. Teachers or students who encounter this puzzle may be able to see that through their engagement with it, they are participating in a considerable tradition of fascination and frustration that goes at least as far back as far as the nineteenth century. The impression, as van Manen (1997) says, may be one of "increased awareness, moral stimulation, insight...[or just] a certain thoughtfulness" (p. 162). Of course, if a study has been especially successful, and the reader particularly receptive, the effect or impression may also be one of "wonder," in which "the taken-for-grantedness of our everyday reality" as van Manen says, is "shattered" (2002).

At the same time, the effects or results of a hermeneutic-phenomenological investigation can sometimes also be stated in practical or even programmatic terms. Van Manen (1997) uses the example of a hypothetical study on childbirth: "health practices may be challenged or changed as a consequence of the increased awareness of the experience of birth by the mother...and father" (p. 162). The above study of the experience of engaging with the Tower of Hanoi in mathematical contexts allows for similarly increased awareness of certain aspects of an experience that might also confirm, or challenge mathematics education or teaching practices. Possible results of this might include the following:

1. The study confirms other research (e.g., Hickey, Moore, & Pellegrino, 2001) that supports the general value of manipulables, or rather of *virtual* manipulables, as ways of engaging students in mathematics, problem solving, and related activities;
2. The study suggests that the emotionally charged character of engagement with the puzzle is an important part of a particular lesson that the puzzle can be used to teach students regarding exponential relations. Students should therefore be given the time and freedom to experience this through the individual engagement demanded by the puzzle in order to learn about or rather, *experience* these particular kinds of mathematical relations; and
3. The study also suggests that this same isolated, emotionally charged engagement has much in common with the experience of mathematical thinking and problem solving more generally. Familiarity with these emotions and with the intense and concrete situations and engagements in which they arise can itself be considered an important part of mathematics education. Students should therefore be encouraged to engage experientially and "emotionally" with puzzles of this kind for this reason as well.

The types of awareness arising from a sustained study of experience are in this sense potentially able to provide insight for mathematics teaching and curriculum in general. At the same time, this kind of research is able to both challenge and confirm the practices and practicalities in everyday educational situations.

This chapter began with a discussion of the role of "experience" as it tends to be understood in both technological design and e-learning research. It showed how the term "experience" is used in studies of online courses as a kind of shorthand for a number of functional and administrative categories such as "student satisfaction" and "student attrition." In addition to the "effects" of "experience" on or for mathematics education, the intervening study of the experience of engaging with the Tower of Hanoi illustrates what can be gained by broadening the understanding of experience in research. Such a broadening suggests that the relationship between experience and education is much more multidimensional than issues like student satisfaction or attrition might initially imply. The depth and vicissitudes of the experiences of those who are learning and exploring online, as this study indicates, readily exceed or "overflow" the predefined categories and distinctions that have been created to differentiate, classify, or otherwise "contain" them. "Tears" or "ecstasy," for example, would hardly make sense as predefined classifications of student experience when working with online mathematics exercises or puzzles. Yet the significance of these kinds of emotional experiences is recognized not only by those who work with these exercises, but also by some of history's most famous mathematicians. Applied to online education and other experiential phenomena, hermeneutic phenomenology holds the promise of being able highlight the existence and significance of intensities of experience and emotion that accompany the intellectual aspects of learning. Such intensities may well be equally important to other subjects and educational processes that may otherwise be excluded from preexisting criteria, measures, and categories.